B Internet Appendix (not for publication)

B.1 Dynamics for a general model

This Section shows how to easily generalize our assumed inflation dynamics for incorporating any number of macroeconomic variables. We assume that the dynamics of a $K \times 1$ vector of risk factors X_t are given by the following vector autoregression:

$$X_{t+1} = \mu + \Phi X_t + v_{t+1}\,, \tag{35}$$

where $v_{t+1} \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma)$. This state vector contains three sets of variables, such that $X_{t+1} = \left(m^*_{t+1},\, \sigma_{t+1},\, y'_{t+1}\right)'$ where m^*_{t+1} and σ_{t+1} are vectors of size $K_m \times 1$ and $K_\sigma \times 1$ respectively and y_t is a vector of yield-specific risk factors of size $K_y \times 1$ ($K = K_m + K_\sigma + K_y$).

Let us consider a set of $K_\chi \times 1$ macroeconomic variables χ_t. We assume that the dynamics of the macroeconomic variables are given by:

$$\chi_t = \overline{\chi} + \Upsilon_m m^*_t + \Upsilon_\sigma \,\mathrm{diag}(\sigma_{t+1})\, \varepsilon^\chi_{t+1}\,, \tag{36}$$

where $\varepsilon^\chi_{t+1} \overset{i.i.d.}{\sim} \mathcal{N}(0, I_{K_\chi})$ and is uncorrelated with v_{t+1}, $\overline{\chi}$ is a $K_\chi \times 1$ vector, Υ_m is a $K_\chi \times K_m$ matrix and Υ_σ is a $K_\chi \times K_\sigma$ matrix.

This formulation still allows to discriminate between macroeconomic trends and volatility factors, which can eventually be of smaller dimension than the number of macroeconomic variables. The remaining pieces of the model write exactly the same as in Section 2 and all pricing and statistical properties remain intact.

B.2 Affine $\mathbb{P}$-property

In this Section, we show that our physical dynamics are affine. Define $u = [u'_x,\, \mathrm{Vec}(U_x)',\, u'_z]'$, where the blocks have respective size K, K^2 and 1. We first introduce the following

Lemma.

Lemma B.1 *The conditional Laplace transform of* $[X_t', \mathrm{Vec}(X_t X_t')']'$ *given its past is given by:*

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(u_x' X_t + X_t' U_x X_t\right) \left| \underline{X_{t-1}}\right.\right] \\
= \; & \exp\left\{ u_x'(I_K - 2\Sigma U_x)^{-1}\left(\mu + \frac{1}{2}\Sigma u_x\right) + \mu' U_x (I_K - 2\Sigma U_x)^{-1}\mu - \frac{1}{2}\log\left|I_K - 2\Sigma U_x\right| \right. \\
+ \; & \left. (u_x + 2U_x\mu)'(I_K - 2\Sigma U_x)^{-1}\Phi X_{t-1} + X_{t-1}\Phi' U_x (I_K - 2\Sigma U_x)^{-1}\Phi X_{t-1} \right\}
\end{aligned}$$

Proof See Cheng and Scaillet (2007). ■

Let us now calculate the conditional Laplace transform of $f_t := [X_t', \mathrm{Vec}(X_t X_t')', \, z_t]$ given $\underline{f_{t-1}}$.

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(u' f_t\right) \left|\underline{f_{t-1}}\right.\right] = \mathbb{E}\left[\exp\left(u_x' X_t + X_t' U_x X_t + u_z z_t\right)\left|\underline{f_{t-1}}\right.\right] \\
= \; & \mathbb{E}\left\{\mathbb{E}\left[\exp\left(u_x' X_t + X_t' U_x X_t + u_z z_t\right)\left|\underline{f_{t-1}}, \, X_t\right.\right]\left|\underline{f_{t-1}}\right.\right\} \\
= \; & \mathbb{E}\left[\exp\left\{u_x' X_t + X_t' U_x X_t + \frac{u_z c}{1 - u_z c}\left[\alpha + \kappa\beta' X_t + X_t'\beta\beta' X_t + \phi z_{t-1}\right]\right\}\left|\underline{f_{t-1}}\right.\right], \\
= \; & \exp\left(\frac{u_z c}{1 - u_z c}(\alpha + \phi z_{t-1})\right)\mathbb{E}\left[\exp\left\{\left(u_x + \frac{\kappa u_z c}{1 - u_z c}\beta\right)' X_t + X_t'\left(U_x + \frac{u_z c}{1 - u_z c}\beta\beta'\right)X_t\right\}\left|\underline{f_{t-1}}\right.\right],
\end{aligned}$$

We hence obtain the conditional Laplace transform of $[X_t', \mathrm{Vec}(X_t X_t')']'$ applied in the two arguments $\left[\left(u_x + \kappa\frac{u_z c}{1-u_z c}\beta\right)'; \mathrm{Vec}\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)'\right]'$. Using Lemma B.1, we have:

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(u' f_t\right)\left|\underline{f_{t-1}}\right.\right] \\
= \; & \exp\left\{\frac{u_z c}{1 - u_z c}(\alpha + \phi z_{t-1}) + \left(u_x + \frac{\kappa u_z c}{1 - u_z c}\beta\right)'\left[I_K - 2\Sigma\left(U_x + \frac{u_z c}{1 - u_z c}\beta\beta'\right)\right]^{-1}\left[\mu + \frac{1}{2}\Sigma\left(u_x + \frac{\kappa u_z c}{1 - u_z c}\beta\right)\right]\right. \\
+ \; & \mu'\left(U_x + \frac{u_z c}{1 - u_z c}\beta\beta'\right)\left[I_K - 2\Sigma\left(U_x + \frac{u_z c}{1 - u_z c}\beta\beta'\right)\right]^{-1}\mu - \frac{1}{2}\log\left|I_K - 2\Sigma\left(U_x + \frac{u_z c}{1 - u_z c}\beta\beta'\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\left(u_x + \frac{\kappa u_z c}{1-u_z c}\beta\right)' + 2\mu'\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right]\left[I_K - 2\Sigma\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right]^{-1}\Phi X_{t-1} \\
&+ X'_{t-1}\Phi'\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\left[I_K - 2\Sigma\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right]^{-1}\Phi X_{t-1}\Bigg\}. \qquad (37)
\end{aligned}
$$

This conditional Laplace transform is an exponential-affine function of f_{t-1}. (f_t) is therefore an affine process under the physical measure. ■

B.3 Convexity adjustment for the pricing kernel

We now turn our interest to the derivation of the convexity adjustment in the pricing kernel given by Equation (6). By no-arbitrage, we have that:

$$
\mathbb{E}_{t-1}(M_t) = e^{-r_{t-1}} \Longleftrightarrow \mathbb{E}_{t-1}\left[\exp\left(\lambda'_{t-1}v_t + \lambda_r \varepsilon_t^r\right)\right] = \exp(\xi_{t-1}) .
$$

To come back to the formulation of Equation (37), it is sufficient to multiply the left- and right-hand side of the previous Equation:

$$
\begin{aligned}
& \mathbb{E}_{t-1}\left[\exp\left(\lambda'_{t-1}v_t + \lambda_r \varepsilon_t^r\right)\right] = \exp(\xi_{t-1}) \\
\Longleftrightarrow \quad & \mathbb{E}_{t-1}\left[\exp\left(\lambda'_{t-1}X_t + \lambda_r z_t\right)\right] = \exp\left[\xi_{t-1} + \lambda'_{t-1}(\mu + \Phi X_{t-1}) + \lambda_r \mathbb{E}_{t-1}(z_t)\right] .
\end{aligned}
$$

Thus, using the result of Equation (37), we obtain:

$$
\begin{aligned}
\xi_{t-1} \;=\; & -\lambda'_{t-1}(\mu + \Phi X_{t-1}) - \lambda_r \mathbb{E}_{t-1}(z_t) + \frac{\lambda_r c}{1-\lambda_r c}(\alpha + \phi z_{t-1}) \\
&+ \left(\lambda_{t-1} + \frac{\kappa \lambda_r c}{1-\lambda_r c}\beta\right)'\left[I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]^{-1}\left[\mu + \frac{1}{2}\Sigma\left(\lambda_{t-1} + \frac{\kappa\lambda_r c}{1-\lambda_r c}\beta\right)\right] \\
&+ \mu'\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\left[I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]^{-1}\mu - \frac{1}{2}\log\left|I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right| \\
&+ \left[\left(\lambda_{t-1} + \frac{\kappa\lambda_r c}{1-\lambda_r c}\beta\right)' + 2\mu'\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]\left[I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]^{-1}\Phi X_{t-1} \\
&+ X'_{t-1}\Phi'\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\left[I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]^{-1}\Phi X_{t-1} .
\end{aligned}
$$

B.4 Multi-horizon Laplace transform

Using the notation:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(u'f_t\right)\left|\underline{f_{t-1}}\right.\right] =: \exp\left\{\mathbb{A}^{\mathbb{Q}}(u) + \mathbb{B}^{\mathbb{Q}'}(u)X_{t-1} + X'_{t-1}\mathbb{C}^{\mathbb{Q}}(u)X_{t-1} + \mathbb{D}^{\mathbb{Q}}(u)z_{t-1}\right\},$$

where:

$$\begin{aligned}
\mathbb{A}^{\mathbb{Q}}(u) &= \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\alpha^{\mathbb{Q}} - \frac{1}{2}\log\left|I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right| \\
&+ \left(u_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\right)'\left[I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right]^{-1}\left[\mu^{\mathbb{Q}} + \frac{1}{2}\Sigma^{\mathbb{Q}}\left(u_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\right)\right] \\
&+ \mu^{\mathbb{Q}'}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\left[I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right]^{-1}\mu^{\mathbb{Q}} \\
\mathbb{B}^{\mathbb{Q}}(u) &= \Phi^{\mathbb{Q}'}\left[I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right]^{-1'}\left[\left(u_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\right) + 2\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)'\mu^{\mathbb{Q}}\right] \\
\mathbb{C}^{\mathbb{Q}}(u) &= \Phi^{\mathbb{Q}'}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\left[I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right]^{-1}\Phi^{\mathbb{Q}} \\
\mathbb{D}^{\mathbb{Q}}(u) &= \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\phi^{\mathbb{Q}}
\end{aligned}$$

Since the one-period ahead conditional risk-neutral Laplace transform of f_t given $\underline{f_{t-1}}$ is exponential-affine in f_{t-1}, it is well-known that the conditional multi-horizon risk-neutral Laplace transform of $(f_t, \ldots, f_{t+k})$ is also exponential-affine in f_{t-1} (see e.g. Darolles, Gourieroux, and Jasiak (2006)). We obtain:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sum_{i=0}^{k} u'_i f_{t+i}\right)\left|\underline{f_{t-1}}\right.\right] &= \exp\Big(\mathbb{A}_k^{\mathbb{Q}}(u_0,\ldots,u_k) + \mathbb{B}_k^{\mathbb{Q}'}(u_0,\ldots,u_k)X_{t-1} \\
&+ X'_{t-1}\mathbb{C}_k^{\mathbb{Q}}(u_0,\ldots,u_k)\,X_{t-1} + \mathbb{D}_k^{\mathbb{Q}}(u_0,\ldots,u_k)\,\mathfrak{r}_{t-1}\Big),
\end{aligned}$$

where:

$$\mathbb{A}_k^{\mathbb{Q}}(u_0,\ldots,u_k) := \mathbb{A}_{k,k}^{\mathbb{Q}}(u_0,\ldots,u_k)$$

$$\begin{aligned}
\mathbb{B}_k^{\mathbb{Q}}(u_0,\ldots,\,u_k) &:= \mathbb{B}_{k,k}^{\mathbb{Q}}(u_0,\ldots,\,u_k)\\
\mathbb{C}_k^{\mathbb{Q}}(u_0,\ldots,\,u_k) &:= \mathbb{C}_{k,k}^{\mathbb{Q}}(u_0,\ldots,\,u_k)\\
\mathbb{D}_k^{\mathbb{Q}}(u_0,\ldots,\,u_k) &:= \mathbb{D}_{k,k}^{\mathbb{Q}}(u_0,\ldots,\,u_k)\,,
\end{aligned}$$

with initial conditions $\mathbb{A}_{k,1}^{\mathbb{Q}}(u_0,\ldots,\,u_k)=\mathbb{A}^{\mathbb{Q}}(u_k)$, $\mathbb{B}_{k,1}^{\mathbb{Q}}(u_0,\ldots,\,u_k)=\mathbb{B}^{\mathbb{Q}}(u_k)$, $\mathbb{C}_{k,1}^{\mathbb{Q}}(u_0,\ldots,\,u_k)=\mathbb{C}^{\mathbb{Q}}(u_k)$ and $\mathbb{D}_{k,1}^{\mathbb{Q}}(u_0,\ldots,\,u_k)=\mathbb{D}^{\mathbb{Q}}(u_k)$, and $\forall i\in\{2,\ldots,k\}$,

$$\begin{aligned}
\mathbb{A}_{k,i}^{\mathbb{Q}}(u_0,\ldots,\,u_k) &= \mathbb{A}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\\
&+ \mathbb{A}^{\mathbb{Q}}\left(u_{k-i+1}+\left[\mathbb{B}_{k,i-1}^{\mathbb{Q}\prime}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{B}_{k,i}^{\mathbb{Q}}(u_0,\ldots,\,u_k) &= \mathbb{B}^{\mathbb{Q}}\left(u_{k-i+1}+\left[\mathbb{B}_{k,i-1}^{\mathbb{Q}\prime}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{C}_{k,i}^{\mathbb{Q}}(u_0,\ldots,\,u_k) &= \mathbb{C}^{\mathbb{Q}}\left(u_{k-i+1}+\left[\mathbb{B}_{k,i-1}^{\mathbb{Q}\prime}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{D}_{k,i}^{\mathbb{Q}}(u_0,\ldots,\,u_k) &= \mathbb{D}^{\mathbb{Q}}\left(u_{k-i+1}+\left[\mathbb{B}_{k,i-1}^{\mathbb{Q}\prime}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}_{k,i-1}^{\mathbb{Q}}(u_0,\ldots,u_k)\right]'\right).
\end{aligned}$$

Since the conditional Laplace transform of f_t given $\underline{f_{t-1}}$ under the physical measure is the same function as the risk-neutral one, but plugging in the physical parameters instead of the risk-neutral ones, we easily obtain:

$$\begin{aligned}
&\varphi_{t-1}(u_0,\ldots,\,u_k)=\mathbb{E}\left[\exp\left(\sum_{i=0}^{k}u_i'f_{t+i}\right)|\underline{f_{t-1}}\right]\\
=:\;&\exp\left(\mathbb{A}_k(u_0,\ldots,\,u_k)+\mathbb{B}_k'(u_0,\ldots,\,u_k)X_{t-1}+X_{t-1}'\mathbb{C}_k(u_0,\ldots,\,u_k)\,X_{t-1}+\mathbb{D}_k(u_0,\ldots,\,u_k)\mathfrak{r}_{t-1}\right)
\end{aligned} \tag{38}$$

where:

$$\begin{aligned}
\mathbb{A}_k(u_0,\ldots,\,u_k) &:= \mathbb{A}_{k,k}(u_0,\ldots,\,u_k)\\
\mathbb{B}_k(u_0,\ldots,\,u_k) &:= \mathbb{B}_{k,k}(u_0,\ldots,\,u_k)\\
\mathbb{C}_k(u_0,\ldots,\,u_k) &:= \mathbb{C}_{k,k}(u_0,\ldots,\,u_k)\\
\mathbb{D}_k(u_0,\ldots,\,u_k) &:= \mathbb{D}_{k,k}(u_0,\ldots,\,u_k)\,,
\end{aligned}$$

with initial conditions $\mathbb{A}_{k,1}(u_0,\ldots,u_k) = \mathbb{A}(u_k)$, $\mathbb{B}_{k,1}(u_0,\ldots,u_k) = \mathbb{B}(u_k)$, $\mathbb{C}_{k,1}(u_0,\ldots,u_k) = \mathbb{C}(u_k)$ and $\mathbb{D}_{k,1}(u_0,\ldots,u_k) = \mathbb{D}(u_k)$, and $\forall i \in \{2,\ldots,k\}$,

$$\begin{aligned}
\mathbb{A}_{k,i}(u_0,\ldots,u_k) &= \mathbb{A}_{k,i-1}(u_0,\ldots,u_k) \\
&+ \mathbb{A}\left(u_{k-i+1} + \left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k), \operatorname{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)', \mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right) \\
\mathbb{B}_{k,i}(u_0,\ldots,u_k) &= \mathbb{B}\left(u_{k-i+1} + \left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k), \operatorname{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)', \mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right) \\
\mathbb{C}_{k,i}(u_0,\ldots,u_k) &= \mathbb{C}\left(u_{k-i+1} + \left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k), \operatorname{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)', \mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right) \\
\mathbb{D}_{k,i}(u_0,\ldots,u_k) &= \mathbb{C}\left(u_{k-i+1} + \left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k), \operatorname{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)', \mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right).
\end{aligned}$$

B.5 Conditional moments of f_t

From Monfort, Renne, and Roussellet (2015) the conditional first two moments of $(X_t', \operatorname{Vec}(X_t X_t')')'$ given the past can be expressed as:

$$\begin{aligned}
\mathbb{E}\left[\begin{pmatrix} X_t \\ \operatorname{Vec}(X_t X_t') \end{pmatrix} \Big| \underline{f_{t-1}}\right] &= \begin{pmatrix} \mu \\ \operatorname{Vec}(\mu\mu' + \Sigma) \end{pmatrix} + \left(\begin{array}{c|c} \Phi & 0 \\ \hline \mu \otimes \Phi + \Phi \otimes \mu & \Phi \otimes \Phi \end{array}\right) \begin{pmatrix} X_{t-1} \\ \operatorname{Vec}(X_{t-1} X_{t-1}') \end{pmatrix} \\
\mathbb{V}\left[\begin{pmatrix} X_t \\ \operatorname{Vec}(X_t X_t') \end{pmatrix} \Big| \underline{f_{t-1}}\right] &= \left(\begin{array}{c|c} \Sigma & \Sigma\Gamma_{t-1}' \\ \hline \Gamma_{t-1}\Sigma & \Gamma_{t-1}\Sigma\Gamma_{t-1}' + (I_{K^2} + \mathcal{K}_K)(\Sigma \otimes \Sigma) \end{array}\right).
\end{aligned}$$

where $\otimes$ is the standard Kronecker product, $\Gamma_{t-1} = [I_K \otimes (\mu + \Phi X_{t-1}) + (\mu + \Phi X_{t-1}) \otimes I_K]$, and $\mathcal{K}_K$ is the $(K^2 \times K^2)$ commutation matrix.

Given that we have conditional first two moments of z_t given f_{t-1} in Appendix A.1, we focus here on the conditional covariance between $(X_t', \operatorname{Vec}(X_t X_t')')'$ and z_t:

$$\mathbb{C}ov\left[\begin{pmatrix} X_t \\ \operatorname{Vec}(X_t X_t') \end{pmatrix}, z_t \Big| \underline{f_{t-1}}\right] = \mathbb{C}ov\left[\begin{pmatrix} X_t \\ \operatorname{Vec}(X_t X_t') \end{pmatrix}, c\left(\alpha + \kappa\beta' X_t + X_t'\beta\beta' X_t + \phi z_{t-1}\right) \Big| \underline{f_{t-1}}\right]$$

$$
= \left(\begin{array}{c} c\Sigma\beta \left[\kappa + 2\left(\mu + \Phi X_{t-1}\right)' \beta\right] \\ \\ c\Gamma_{t-1}\Sigma\beta \left[\kappa + 2\left(\mu + \Phi X_{t-1}\right)' \beta\right] + 2c\mathrm{Vec}(\Sigma\beta\beta'\Sigma) \end{array} \right).
$$

In the end, putting the previous results together, we obtain that f_t follows a semi-strong VAR of the form

$$
f_t = \Psi_0 + \Psi f_{t-1} + \mathrm{Vec}^{-1/2}(\Omega_0 + \Omega f_{t-1})\zeta_t \tag{39}
$$

with parameters given by:

$$
\Psi_0 = \left(\begin{array}{c} \mu \\ \\ \mathrm{Vec}(\mu\mu' + \Sigma) \\ \\ c\left[\alpha + \left(\kappa + \beta'\mu\right)\beta'\mu + \beta'\Sigma\beta\right] \end{array} \right),
$$

$$
\Psi = \left(\begin{array}{c|c|c} \Phi & 0 & 0 \\ \hline \mu \otimes \Phi + \Phi \otimes \mu & \Phi \otimes \Phi & 0 \\ \hline c\left(\kappa + 2\mu'\beta\right)\beta'\Phi & c\left(\beta'\Phi \otimes \beta'\Phi\right) & c\phi \end{array} \right),
$$

and,

$$
\mathrm{Vec}^{-1}(\Omega_0 + \Omega f_{t-1}) =
$$

$$
\left(\begin{array}{c|c|l} \Sigma & \Sigma\Gamma'_{t-1} & c\Sigma\beta\left[\kappa + 2\left(\mu + \Phi X_{t-1}\right)'\beta\right] \\ \hline & \Gamma_{t-1}\Sigma\Gamma'_{t-1} + (I_{K^2} + \mathcal{K}_K)(\Sigma \otimes \Sigma) & c\Gamma_{t-1}\Sigma\beta\left[\kappa + 2\left(\mu + \Phi X_{t-1}\right)'\beta\right] + 2c\mathrm{Vec}(\Sigma\beta\beta'\Sigma) \\ \hline & & c^2\left(\left[\kappa + 2\left(\mu + \Phi X_{t-1}\right)'\beta\right]^2 \beta'\Sigma\beta + 2\left(\beta'\Sigma\beta\right)^2\right) \\ & & + \quad 2c^2\left(\alpha + \phi z_{t-1} + \left(\kappa + \beta'\mu\right)\beta'\mu + \beta'\Sigma\beta + \left(\kappa + 2\mu'\beta\right)\beta'\Phi X_{t-1} + \left(\beta'\Phi X_{t-1}\right)^2\right). \end{array} \right)
$$

Looking at the Ψ matrix, we can easily see that the system is stationary as soon as the eigenvalues of Φ are inside the unit circle and that $c\phi$ is below one. In this case, $(I_{K+K^2+1}-\Psi)^{-1}$ exists, and we have:[13]

$$\begin{aligned}
\mathbb{E}\left(f_{t+n}|\underline{f_t}\right) &= \mathbb{E}\left(\Psi_0+\Psi f_{t+n-1}|\underline{f_t}\right)\\
&= \sum_{i=0}^{n-1}\Psi^i\Psi_0+\Psi^n f_t
\end{aligned}$$

Noting that:

$$\sum_{i=0}^{n-1}\Psi^i=(I_{K+K^2+1}-\Psi)^{-1}(I_{K+K^2+1}-\Psi^n)\,.$$

we obtain:

$$\begin{aligned}
\mathbb{E}\left(f_{t+n}|\underline{f_t}\right) &= (I_{K+K^2+1}-\Psi)^{-1}(I_{K+K^2+1}-\Psi^n)\Psi_0+\Psi^n f_t\\
\mathbb{E}(f_t) &= (I_{K+K^2+1}-\Psi)^{-1}\Psi_0\,.
\end{aligned}$$

For the conditional variance, we apply the law of total variance and obtain:

$$\begin{aligned}
\mathbb{V}\left(f_{t+n}|\underline{f_t}\right) &= \mathbb{V}\left[\mathbb{E}\left(f_{t+n}|\underline{f_{t+n-1}}\right)|\underline{f_t}\right]+\mathbb{E}\left[\mathbb{V}\left(f_{t+n}|\underline{f_{t+n-1}}\right)|\underline{f_t}\right]\\
&= \mathbb{V}\left(\Psi f_{t+n-1}|\underline{f_t}\right)+\mathbb{E}\left[\mathrm{Vec}^{-1}(\Omega_0+\Omega f_{t+n-1})|\underline{f_t}\right]\\
&= \Psi\mathbb{V}\left(f_{t+n-1}|\underline{f_t}\right)\Psi'+\mathrm{Vec}^{-1}\left[\Omega_0+\Omega\mathbb{E}\left(f_{t+n-1}|\underline{f_t}\right)\right]\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Vec}\left[\mathbb{V}\left(f_{t+n}|\underline{f_t}\right)\right] &= (\Psi\otimes\Psi)\mathrm{Vec}\left[\mathbb{V}\left(f_{t+n-1}|\underline{f_t}\right)\right]+\\
&\quad\left[\Omega_0+\Omega\left\{(I_{K+K^2+1}-\Psi)^{-1}\left(I_{K+K^2+1}-\Psi^{n-1}\right)\Psi_0+\Psi^{n-1}f_t\right\}\right]\,.
\end{aligned}$$

A simple recursion gives the conditional variance as of function of the current value

13. The conditional moments formulas are given with the use of the matrix $(I_{K+K^2+1}-\Psi)^{-1}$ which is only invertible if the system is stationary. Note that the stationarity assumption is however not necessary and the same formulas can be expressed in the form of truncated sums.

of the factors:

$$\text{Vec}\left[\mathbb{V}\left(f_{t+n}\big|\underline{f_t}\right)\right] = \sum_{i=0}^{n-1}(\Psi\otimes\Psi)^i\left(\Omega_0+\Omega\left[(I_{K+K^2+1}-\Psi)^{-1}(I_{K+K^2+1}-\Psi^{n-i-1})\Psi_0+\Psi^{n-i-1}f_t\right]\right)\ .$$

For the marginal variance, again using the law of total variance we have:

$$\begin{aligned}\mathbb{V}\left(f_t\right) &= \mathbb{V}\left[\mathbb{E}\left(f_t|\underline{f_{t-1}}\right)\right]+\mathbb{E}\left[\mathbb{V}\left(f_t|\underline{f_{t-1}}\right)\right] \\ &= \Psi\mathbb{V}\left(f_t\right)\Psi'+\text{Vec}^{-1}\left[\Omega_0+\Omega\mathbb{E}\left(f_t\right)\right]\ ,\end{aligned}$$

Both first and second order conditional and unconditional moments of the extended vector f_t are thus closed form functions.

B.6 The Quadratic Kalman Filter

As in Cheng and Scaillet (2007), we stack together the linear and quadratic components of the risk factors. We denote by:

$$f_t^{(aug)} = \left(X_t^{(aug)'},\, X_t^{(aug)'}\otimes X_t^{(aug)'},\, z_t\right)'$$

We have shown that $f_t^{(aug)}$ is an affine process thus it possesses a semi-strong VAR form. Stacking together the transition and the measurement equations, we obtain:

$$\begin{aligned} f_t^{(aug)} &= \Psi_0^{(aug)}+\Psi^{(aug)}f_{t-1}^{(aug)}+\left[\text{Vec}^{-1}\left(\Omega_0^{(aug)}+\Omega^{(aug)}f_{t-1}^{(aug)}\right)\right]^{1/2}\zeta_t^{(aug)} \\ \mathcal{Y}_t^{(obs)} &=: \mathcal{A}+\widetilde{\mathcal{B}}'f_t^{(aug)}+\eta_t\,, \end{aligned} \tag{40}$$

where $\mathcal{A}$ and $\mathcal{B}$ stack respectively the intercepts and the loadings of the different observables. Notice that the transition Equation has been modified with respect to the one presented in Appendix B.5 because of the addition of the idiosyncratic inflation shock ε_t^π to the system. Since extending the system is straightforward, we do not detail it here.

The Quadratic Kalman Filter (QKF) is particularly fitted to this class of models. The original filtering algorithm has been applied to state-space models where the transition dynamics are given by a Gaussian VAR and the measurement equations are linear-quadratic. This algorithm is slightly modified to incorporate z_t (which is non-Gaussian) and is detailed below.

Since the state-space model expressed with respect to f_t is affine, we can apply the Kalman filter algorithm. Using the notations $f_{t|t-1}^{(aug)} = \mathbb{E}(f_t^{(aug)}|\underline{\mathcal{Y}_{t-1}^{(obs)}})$, $P_{t|t-1} = \mathbb{V}(f_t^{(aug)}|\underline{\mathcal{Y}_{t-1}^{(obs)}})$, $f_{t|t}^{(aug)} = \mathbb{E}(f_t^{(aug)}|\underline{\mathcal{Y}_t^{(obs)}})$, $\mathcal{Y}_{t|t-1}^{(obs)} = \mathbb{E}(\mathcal{Y}_t^{(obs)}|\underline{\mathcal{Y}_{t-1}^{(obs)}})$, $M_{t|t-1} = \mathbb{V}(\mathcal{Y}_t^{(obs)}|\underline{\mathcal{Y}_{t-1}^{(obs)}})$, $P_{t|t} = \mathbb{V}(f_t^{(aug)}|\underline{\mathcal{Y}_t^{(obs)}})$, the steps in the algorithm are the following. Initialize the filter at $f_{0|0}^{(aug)} = \mathbb{E}(f_t^{(aug)})$ and $P_{0|0} = \mathbb{V}(f_t^{(aug)})$ using the results of Appendix B.5. Then, for each period t, predict the latent:

$$\begin{aligned} f_{t|t-1}^{(aug)} &= \Psi_0^{(aug)} + \Psi^{(aug)} f_{t-1|t-1}^{(aug)} \\ P_{t|t-1} &= \Psi^{(aug)} P_{t-1|t-1} \Psi^{(aug)\prime} + \text{Vec}^{-1}\left(\Omega_0^{(aug)} + \Omega^{(aug)} f_{t-1|t-1}^{(aug)}\right), \end{aligned}$$

predict the observable:

$$\begin{aligned} \mathcal{Y}_{t|t-1}^{(obs)} &= \mathcal{A} + \widetilde{\mathcal{B}}' f_{t|t-1}^{(aug)} \\ M_{t|t-1} &= \widetilde{\mathcal{B}}' P_{t|t-1} \widetilde{\mathcal{B}} + \mathbb{V}(\eta_t), \end{aligned}$$

update the prediction of the latent:

$$\begin{aligned} f_{t|t}^{(aug)} &= f_{t|t-1}^{(aug)} + P_{t|t-1} \widetilde{\mathcal{B}} M_{t|t-1}^{-1} \left(\mathcal{Y}_t^{(obs)} - \mathcal{Y}_{t|t-1}^{(obs)}\right) \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1} \widetilde{\mathcal{B}} M_{t|t-1}^{-1} \widetilde{\mathcal{B}}' P_{t|t-1}, \end{aligned}$$

and compute the quasi log-likelihood assuming that the conditional distribution of $\mathcal{Y}_t^{(obs)}$ given $\underline{\mathcal{Y}_{t-1}^{(obs)}}$ is Gaussian with mean $\mathcal{Y}_{t|t-1}^{(obs)}$ and variance $M_{t|t-1}$.

$$\mathcal{L}_t = -\frac{1}{2}\left[K_{obs}\log(2\pi) + \log|M_{t|t-1}| + \left(\mathcal{Y}_t^{(obs)} - \mathcal{Y}_{t|t-1}^{(obs)}\right)' M_{t|t-1}^{-1} \left(\mathcal{Y}_t^{(obs)} - \mathcal{Y}_{t|t-1}^{(obs)}\right)\right].$$

where K_{obs} is the dimension of the vector $\mathcal{Y}_t^{(obs)}$. In order to be consistent with the theoretical properties of the processes, two corrections are applied to the filtered values after storing the results. First, if the components of $z_{t|t}$ are negative, they are set to zero. Second, the filtered values of $\left(X^{(aug)} \otimes X^{(aug)}\right)_{t|t}$ are imposed to be exactly equal to $\left(X_{t|t}^{(aug)} \otimes X_{t|t}^{(aug)}\right)$.

As for the standard Kalman filter, the QKF provides a convenient way to handle missing data. One just has to adjust the size of the parameters in the measurement equations to predict only the variables that are observed. The measurement equation rewrites

$$\widetilde{\mathcal{Y}}_t^{(obs)} = E_t \left(\mathcal{A} + \widetilde{\mathcal{B}}' f_t^{(aug)} + \eta_t\right) =: \mathcal{A}_t + \widetilde{\mathcal{B}}_t' f_t^{(aug)} + E_t \eta_t \,,$$

where $\widetilde{\mathcal{Y}}_t^{(obs)}$ is the subset of variables of Y_t that is observed, and E_t is a matrix selecting the corresponding rows. The prediction and update states remain the same using the adjusted parameters.

B.7 Primary Dealer Survey data

The primary dealer surveys (PDS) are publicly available from January 2011 on. They are conducted by the New York Fed to inform the FOMC members of primary dealer's expectation about the economy, monetary policy and financial markets developments. They are conducted on a regular basis, prior to the FOMC meetings (8 per year) in January, March, April, June, July, September, October and December.[14] The questions and statistics collected have evolved to adapt to the economic environment, which makes it difficult to create homogeneous time-series on the probability to stay at the zero lower bound for a year.

We construct the conditional probabilities of staying at zero for a year using the ques-

14. See the survey results on https://www.newyorkfed.org/markets/primarydealer_survey_questions.html.

tion: *Of the possible outcomes below, please indicate the percent chance you attach to the timing of the first federal funds target rate increase.* (question #2 of each survey). The answer takes the form of a table associating the average of all participant answers per horizon. Table 5 provides two examples.

Table 5: Examples of primary dealer survey answers

Panel(a): January 2011

	2011				2012				$\geqslant$ 2013
	Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4	Q1
Average	0%	1%	2%	11%	14%	13%	16%	17%	25%

Panel(b): March 2013

	2013		2014		2015		2016		2017	
	H1	H2	H1	H2	H1	H2	H1	H2	H1	$\geqslant$H2
Average	0%	1%	5%	10%	23%	27%	18%	8%	4%	4%

As can be seen on Table 5, the horizons of the question can be for next quarter or next semester. For all time periods where the horizons are quarterly or below, we aggregate the answers to get semi-annual horizons for homogeneity. We then compute the probabilities as follow. Let $\mathcal{M}_t = \{1, \dots, 12\}$ be the number of the current date-t month, $\mathcal{Y}_t$ the number of date-t year, and $\mathcal{H}_t = 1 + \mathbb{1}\{\mathcal{M}_t \in \{7, \dots, 12\}\}$ the indicator of the semester. Let $p_t(\mathcal{H}_t, \mathcal{Y}_t)$ be the answer given in the survey. Our probabilities are given by:

$$
\begin{aligned}
&[1 - p_t(\mathcal{H}_t, \mathcal{Y}_t)] \times [1 - p_t(\mathcal{H}_t + 1, \mathcal{Y}_t)] \times \left[1 - p_t(\mathcal{H}_t, \mathcal{Y}_t + 1)\,\tfrac{\mathcal{M}_t - 1}{12}\right] \quad \text{if} \quad \mathcal{H}_t = 1 \\
&[1 - p_t(\mathcal{H}_t, \mathcal{Y}_t)] \times [1 - p_t(\mathcal{H}_t - 1, \mathcal{Y}_t + 1)] \times \left[1 - p_t(\mathcal{H}_t, \mathcal{Y}_t + 1)\,\tfrac{\mathcal{M}_t - 1}{12}\right] \quad \text{if} \quad \mathcal{H}_t = 2\,.
\end{aligned}
$$

The previous formula assumes that inside the last semester considered, the timing of the first increase is uniformly distributed. For example, for the two panels of Table 5, we obtain the probabilities:

$$
\begin{aligned}
2011-01 &\rightarrow [1-(0+0.01)] \times [1-(0.02+0.11)] \simeq 0.86 \\
2013-03 &\rightarrow (1-0) \times (1-0.01) \times \left(1-\frac{3-1}{12}0.05\right) \simeq 0.965\,.
\end{aligned}
$$

Last, in order to avoid the fitted series to be too volatile, we fill out the missing data with the last available data point (step function) and impose that the measurement errors standard deviation is equal to 15% of the obtained series standard deviation.

B.8 Campbell-Shiller regression coefficients

The excess returns of any bond for k-holding periods can be defined as the return of a strategy consisting in buying the bond at time t and selling it at time $t+k$, minus the risk-less interest rate of maturity k. This k-period risk-less rate is equal to $R_t^{(k)}$ in the nominal world and $R_t^{(k)*}$ in the real world.

Proposition B.1 *The k-period nominal excess returns of nominal bonds and real excess returns of TIPS of maturity n are written:*

$$
\mathcal{XR}_{t+k} = \frac{n-k}{k}\left[R_t^{(n)} - R_{t+k}^{(n-k)}\right] + R_t^{(n)} - R_t^{(k)} \quad (41)
$$

$$
\mathcal{XR}^*_{t+k} = \frac{n-k}{k}\left[R_t^{(n)*} - R_{t+k}^{(n-k)*}\right] + R_t^{(n)*} - R_t^{(k)*}\,. \quad (42)
$$

Corollary B.1.1 *The nominal and real expected excess returns of nominal bonds and TIPS at date t are affine functions of f_t computable in closed-form.*

Indeed, the one-year excess returns of holding a nominal bond of maturity n are given by:

$$
\frac{1}{k}\log\left(\frac{P_{t+k}^{(n-k)}}{P_t^{(n)}}\right) - R_t^{(k)} = \frac{n}{k}R_t^{(n)} - \frac{n-k}{k}R_{t+k}^{(n-k)} - R_t^{(k)}\,.
$$

For the excess returns of TIPS, I denote by $P^{(n-k)*}_{t\to t+k}$ the price at $t+k$ of the TIPS issued at time t of maturity n. Let CPI_t be the reference inflation index at date t.

$$P^{(n-k)*}_{t\to t+k} = \mathbb{E}\left[M_{t+k,t+n}\frac{\text{CPI}_{t+n}}{\text{CPI}_t}|\underline{f_{t+k}}\right], \tag{43}$$

where the principal is adjusted by the reference price-index variation between the inception and the maturity date (t and $t+n$). Rearranging formula (43), this price can be expressed with the price of a newly issued TIPS at date $t+k$.

$$P^{(n-k)*}_{t\to t+k} = \mathbb{E}\left[M_{t+k,t+n}\frac{\text{CPI}_{t+n}}{\text{CPI}_{t+k}}|\underline{f_{t+k}}\right]\frac{\text{CPI}_{t+k}}{\text{CPI}_t} = P^{(n-k)*}_{t+k}\exp(\pi_{t:t+k}), \tag{44}$$

where $\pi_{t:t+k}$ is the inflation rate between t and $t+k$. Therefore, the excess returns of holding TIPS for k-holding periods are given by:

$$\begin{aligned}\frac{1}{k}\log\left(\frac{P^{(n-k)*}_{t+k}}{P^{(n)*}_t}\exp(\pi_{t:t+k})\right) - \frac{1}{k}\pi_{t:t+k} - R^{(k)*}_t &= \frac{1}{k}\log\left(\frac{P^{(n-k)*}_{t+k}}{P^{(n)*}_t}\right) - R^{(k)*}_t \\ &= \frac{n}{k}R^{(n)*}_t - \frac{n-k}{k}R^{(n-k)*}_{t+k} - R^{(k)*}_t .\end{aligned}$$

These excess returns computations can be used to test whether the model is able to reproduce the deviations from the expectation hypothesis consistently with the data, and whether the model-implied predictions of excess returns are reasonable. These two tests are respectively called LPY-I and LPY-II in the terminology of Dai and Singleton (2002). Both LPY-I and LPY-II reformulates the excess returns in the form of the well-known Campbell and Shiller (1991) regressions (CS henceforth).

Proposition B.2 *The CS regressions are given by:*[15]

$$R^{(n-k)}_{t+k} - R^{(n)}_t = \alpha_{k,n} + \beta_{k,n}\frac{k}{n-k}\left(R^{(n)}_t - R^{(k)}_t\right) + \epsilon_{t+k,n} \tag{45}$$

15. The same formulations can be found in Haubrich et al. (2012). Evans (1998) formulates a slightly different regression with the Equation (20) of his paper. He expresses the expectation hypothesis equating the expected nominal excess returns of TIPS with the expected nominal excess returns of nominal bonds.

$$R_{t+k}^{(n-k)*} - R_t^{(n)*} \quad = \quad \alpha_{k,n}^* + \beta_{k,n}^* \frac{k}{n-k}\left(R_t^{(n)*} - R_t^{(k)*}\right) + \epsilon_{t+k,n}^* \,. \tag{46}$$

All model-implied intercepts and slopes $\alpha_{k,n}$, $\alpha_{k,n}^*$, $\beta_{k,n}$, and $\beta_{k,n}^*$ are computable in closed-form.

Due to the similarities of the two specifications, we only present the computations of the coefficients for (45). By the properties of OLS estimator, the optimal slope of this regression is given by:

$$\beta_{k,n} = \frac{n-k}{k} \times \frac{\mathbb{C}ov\left[R_{t+k}^{(n-k)} - R_t^{(n)}, R_t^{(n)} - R_t^{(k)}\right]}{\mathbb{V}\left[R_t^{(n)} - R_t^{(k)}\right]} \,.$$

Using the notation $f_t = (X_t', \, X_t' \otimes X_t', \, z_t)'$,

$$R_t^{(n)} = \mathcal{A}_n + \mathcal{B}_n^{(R)'} f_t \,,$$

we obtain:

$$\beta_{k,n} \quad = \quad \frac{n-k}{k} \times \frac{\mathbb{C}ov\left[\mathcal{B}_{n-k}^{(R)'} f_{t+k} - \mathcal{B}_n^{(R)'} f_t, \mathcal{B}_n^{(R)'} f_t - \mathcal{B}_k^{(R)'} f_t\right]}{\mathbb{V}\left[\mathcal{B}_n^{(R)'} f_t - \mathcal{B}_k^{(R)'} f_t\right]} \,,$$

which can be simplified using the semi-strong VAR form of Equation (39):

$$= \quad \frac{n-k}{k} \times \frac{\mathbb{C}ov\left[\left(\mathcal{B}_{n-k}^{(R)'} \Psi^k - \mathcal{B}_n^{(R)'}\right) f_t, \left(\mathcal{B}_n^{(R)'} - \mathcal{B}_k^{(R)'}\right) f_t\right]}{\mathbb{V}\left[\left(\mathcal{B}_n^{(R)'} - \mathcal{B}_k^{(R)'}\right) f_t\right]}$$

$$= \quad \frac{n-k}{k} \times \frac{\left[\left(\mathcal{B}_n^{(R)'} - \mathcal{B}_k^{(R)'}\right) \otimes \left(\mathcal{B}_{n-k}^{(R)'} \Psi^k - \mathcal{B}_n^{(R)'}\right)\right] \left(I_{(K+K^2+1)^2} - (\Psi \otimes \Psi)\right)^{-1} \left(\Omega_0 + \Omega \mathbb{E}(f_t)\right)}{\left[\left(\mathcal{B}_n^{(R)'} - \mathcal{B}_k^{(R)'}\right) \otimes \left(\mathcal{B}_n^{(R)'} - \mathcal{B}_k^{(R)'}\right)\right] \left(I_{(K+K^2+1)^2} - (\Psi \otimes \Psi)\right)^{-1} \left(\Omega_0 + \Omega \mathbb{E}(f_t)\right)} \,,$$

where $\mathbb{E}(f_t) = (I_{K+K^2+1} - \Psi)^{-1}\Psi_0$. The proofs for the other regressions are of similar fashion, since all dependent and independent variables of all regressions can be expressed as affine functions of the process f_t.

If the expectation hypothesis was holding true, intercept and slopes would all be respectively equal to 0 and 1 and the corresponding excess return would average to zero. However, since the expectation hypothesis is largely violated in practice, the current slope of nominal/real interest rates can predict future excess returns. In practice, we consider $k = 12$ months. Testing LPY-I consists in estimating regressions (45-46) on the data for maturities ranging from 1 to 10 years, and comparing the estimated regression coefficients to the model-implied ones.[16] Testing LPY-II consists in performing the same regressions on the data adding the corresponding model-implied expected excess returns series on the right-hand side of the regression. Adding the expected excess return should in theory correct the deviations from the expectation hypothesis.[17] A consistent model should be able to produce $\beta_{k,n}$ coefficients non significantly different from 1.

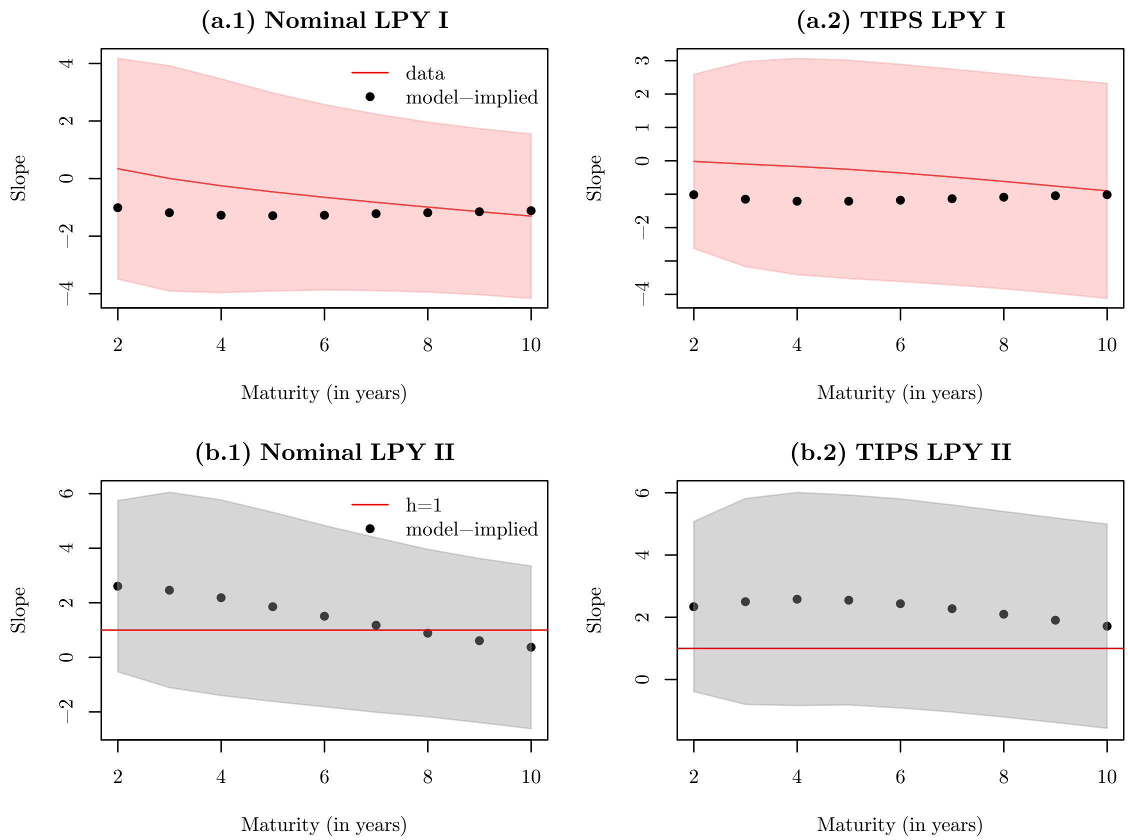
The results presented in Figure 11 show that all model-implied regression slope estimates are not statistically different (5% level) from the empirical estimates obtained directly from the data. We also perform joint F-tests considering not only the slopes of the regression but also the intercepts at the same time. Estimates gathered in Table 6 show that for all maturities we are not able to reject (I) the equality between model-implied and data-based intercepts and slopes in the first set of regressions, and (II) the equality of intercepts and slopes to (0,1) in the second set of regressions.

B.9 Additional Tables and Figures from the benchmark ELB model

16. To obtain the yields of nominal bonds and TIPS at all maturities for the whole sample period, we use the model-implied yield series reconstructed from the filtered factors and omit the measurement errors.

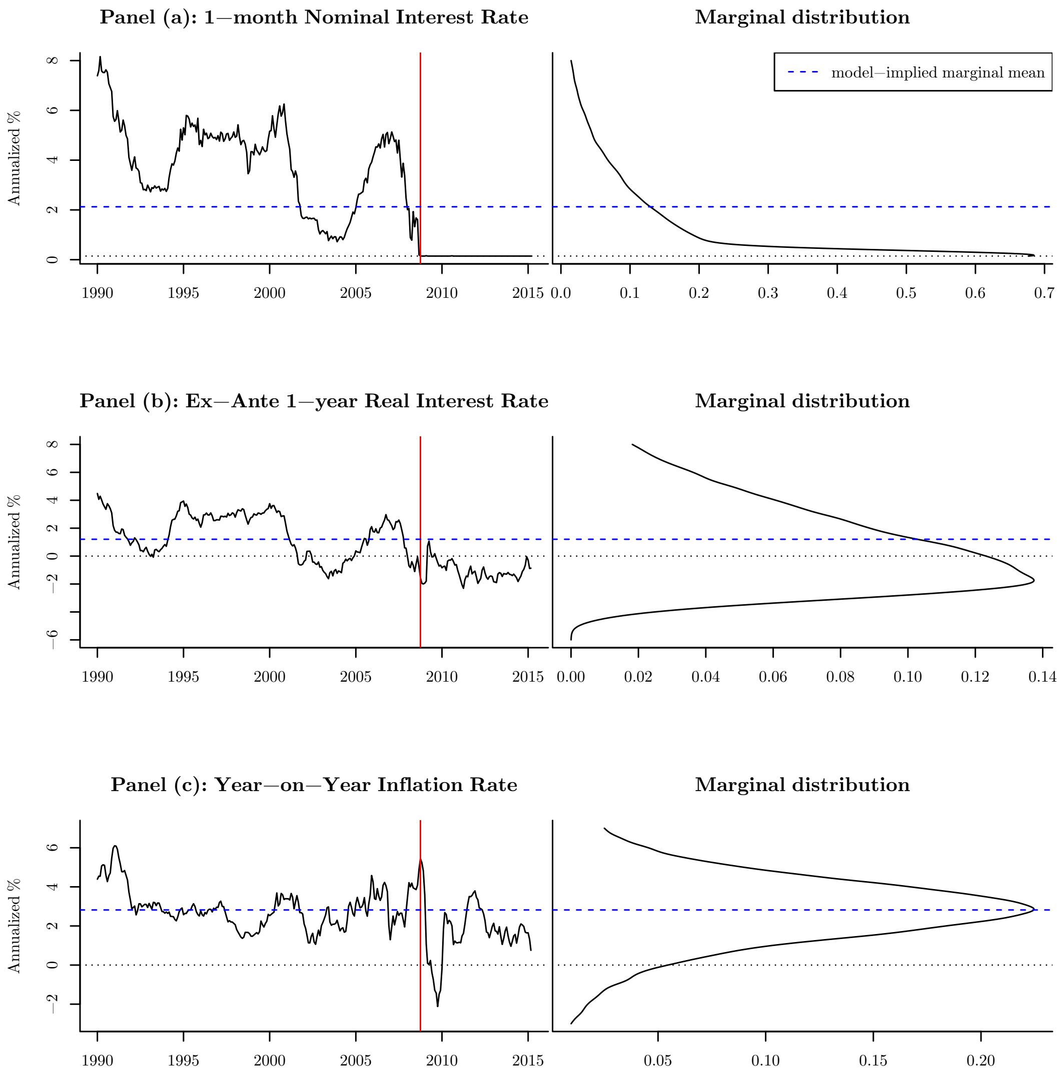
17. We add the series of expected excess returns to the regressor so that we still estimate one regression slope.

Figure 11: Campbell-Shiller regression slopes



Notes: These graphs present the slopes of Campbell and Shiller regressions with a 12-months holding period. On panel (a), the red solid line gathers the slope estimates obtained with yields data. 95% Confidence intervals are computed using Newey-West robust estimators with automatically selected lag and are indicated with the red-shaded areas. Model-implied estimates are indicated with the black dots and computed with the yields and inflation expectation and variance formulas. On panel (b), the red solid line represents the theoretical values of the regression, namely one for all maturities. Model-implied estimates are indicated with the black dots and computed performing the Campbell and Shiller regressions where the dependent variable is adjusted by the model-implied expected excess returns. 95% Confidence intervals are computed using Newey-West robust estimators with automatically selected lag and are indicated with the grey-shaded areas.

Figure 12: Distribution of short-term interest rates and inflation



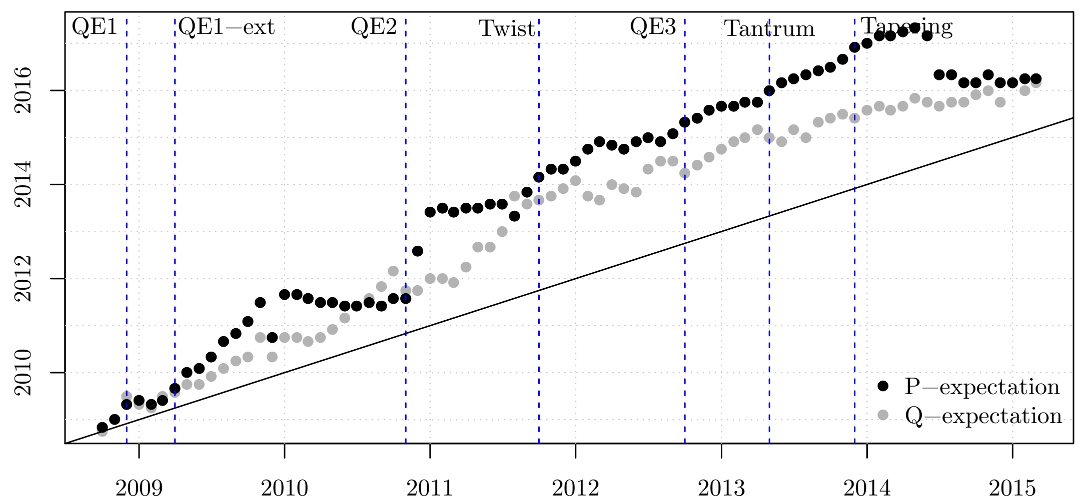
Notes: Marginal distributions on the right-hand side are computed with kernel density estimates on 100,000 simulations.

Table 6: Campbell-Shiller regressions: 12 months holding period

	Panel (I): LPY-I regressions									
	Maturity (yrs)	2	3	4	5	6	7	8	9	10
Nominal bonds	$\hat{\alpha}_{12,n}^{(\text{model})}$	-0.341	-0.228	-0.231	-0.239	-0.235	-0.223	-0.208	-0.193	-0.179
	$\hat{\alpha}_{12,n}^{(\text{data})}$	-0.659	-0.548	-0.462	-0.385	-0.318	-0.260	-0.211	-0.169	-0.132
	$\sigma_{\text{NW}}\left(\hat{\alpha}_{12,n}^{(\text{data})}\right)$	(0.854)	(0.730)	(0.646)	(0.577)	(0.521)	(0.476)	(0.438)	(0.408)	(0.384)
	$\hat{\beta}_{12,n}^{(\text{model})}$	-1.011	-1.178	-1.266	-1.289	-1.268	-1.227	-1.185	-1.149	-1.122
	$\hat{\beta}_{12,n}^{(\text{data})}$	0.343	0.006	-0.248	-0.461	-0.649	-0.824	-0.989	-1.148	-1.302
	$\sigma_{\text{NW}}\left(\hat{\beta}_{12,n}^{(\text{data})}\right)$	(1.957)	(1.997)	(1.895)	(1.755)	(1.644)	(1.564)	(1.506)	(1.472)	(1.458)
	Joint p-val	0.78	0.84	0.86	0.88	0.91	0.94	0.98	0.99	0.99
Tips	$\hat{\alpha}_{12,n}^{*(\text{model})}$	-0.283	-0.169	-0.169	-0.178	-0.178	-0.170	-0.160	-0.150	-0.140
	$\hat{\alpha}_{12,n}^{*(\text{data})}$	-0.372	-0.366	-0.355	-0.331	-0.301	-0.268	-0.236	-0.206	-0.178
	$\sigma_{\text{NW}}\left(\hat{\alpha}_{12,n}^{*(\text{data})}\right)$	(0.479)	(0.461)	(0.447)	(0.432)	(0.413)	(0.393)	(0.374)	(0.356)	(0.341)
	$\hat{\beta}_{12,n}^{*(\text{model})}$	-1.013	-1.152	-1.208	-1.210	-1.176	-1.131	-1.085	-1.047	-1.017
	$\hat{\beta}_{12,n}^{*(\text{data})}$	-0.016	-0.097	-0.169	-0.255	-0.361	-0.483	-0.616	-0.756	-0.898
	$\sigma_{\text{NW}}\left(\hat{\beta}_{12,n}^{*(\text{data})}\right)$	(1.331)	(1.567)	(1.653)	(1.668)	(1.660)	(1.648)	(1.641)	(1.638)	(1.643)
	Joint p-val	0.75	0.79	0.82	0.85	0.89	0.92	0.96	0.98	0.99
	Panel (II): LPY-II regressions									
Nominal bonds	$\hat{\alpha}_{12,n}$	-0.348	-0.554	-0.566	-0.513	-0.442	-0.370	-0.305	-0.249	-0.200
	$\sigma_{\text{NW}}\left(\hat{\alpha}_{12,n}\right)$	(0.668)	(0.663)	(0.627)	(0.580)	(0.527)	(0.478)	(0.425)	(0.389)	(0.361)
	$\hat{\beta}_{12,n}$	2.608	2.470	2.186	1.850	1.509	1.187	0.890	0.617	0.366
	$\sigma_{\text{NW}}\left(\hat{\beta}_{12,n}\right)$	(1.601)	(1.826)	(1.8282	(1.769)	(1.694)	(1.632)	(1.565)	(1.534)	(1.521)
	Joint p-val.	0.59	0.68	0.66	0.63	0.59	0.53	0.45	0.39	0.34
Tips	$\hat{\alpha}_{12,n}^{*}$	0.162	-0.174	-0.320	-0.380	-0.393	-0.382	-0.361	-0.334	-0.307
	$\sigma_{\text{NW}}\left(\hat{\alpha}_{12,n}^{*}\right)$	(0.551)	(0.571)	(0.539)	(0.492)	(0.465)	(0.431)	(0.399)	(0.370)	(0.345)
	$\hat{\beta}_{12,n}^{*}$	2.346	2.508	2.587	2.556	2.444	2.282	2.097	1.905	1.714
	$\sigma_{\text{NW}}\left(\hat{\beta}_{12,n}^{*}\right)$	(1.394)	(1.688)	(1.746)	(1.721)	(1.713)	(1.697)	(1.685)	(1.676)	(1.673)
	Joint p-val.	0.25	0.58	0.65	0.66	0.67	0.67	0.66	0.63	0.60

Notes: This table presents the results of several Campbell-Shiller regressions for LPY-I and LPY-II conditions for excess returns of 12-months holding period (see Dai and Singleton (2002) and Technical Appendix B.8). Panel (I) presents values of intercepts and slopes for LPY-I regressions computed with model-implied parameters (subscript *model*) and with OLS on fitted data (subscript *data*). The joint p-value is the p-value associated with the F-statistic testing whether model-implied and data-implied quantities are equal. Panel (II) presents values of intercepts and slopes for LPY-II regressions computed with OLS on fitted data when the dependent variable is corrected from model-implied expected excess returns. The joint p-value is the p-value associated with the F-statistic testing whether intercepts and slopes are equal to $(0, 1)$.

Figure 13: Expected liftoff date



Notes: The black and grey dots represent respectively the physical and risk-neutral expectations of the future liftoff date. The black solid line is the 45 degree line. The blue dashed lines are the different unconventional monetary policy episodes, namely: QE1, QE1-extension, QE2, Operation Twist, QE3, Taper tantrum, and the Tapering.

B.10 QTSM estimates and results

Table 7: Parameter estimates: X_t dynamics: Standard QTSM

	estimates	std.		estimates	std.
μ_{π^*}	0.0037	(0.0089)	$\mu^{\mathbb{Q}}_{\pi^*}$	0.0037	(0.0089)
μ_{σ}	0	–	$\mu^{\mathbb{Q}}_{\sigma}$	0	–
μ_{y_1}	0	–	$\mu^{\mathbb{Q}}_{y_1}$	0	–
μ_{y_2}	-0.3102^{***}	(0.0703)	$\mu^{\mathbb{Q}}_{y_2}$	-0.3102^{***}	(0.0703)
Φ_{π^*}	0.8724^{***}	(0.0177)	$\Phi^{\mathbb{Q}}_{\pi^*}$	0.9671^{***}	(0.0014)
Φ_{σ,π^*}	0	–	$\Phi^{\mathbb{Q}}_{\sigma,\pi^*}$	0	–
Φ_{y_1,π^*}	0	–	$\Phi^{\mathbb{Q}}_{y_1,\pi^*}$	0	–
Φ_{y_2,π^*}	0	–	$\Phi^{\mathbb{Q}}_{y_2,\pi^*}$	-0.0378^{***}	(0.0027)
$\Phi_{\pi^*,\sigma}$	-0.1136^{***}	(0.0236)	$\Phi^{\mathbb{Q}}_{\pi^*,\sigma}$	-0.1136^{***}	(0.0236)
Φ_{σ}	0.9922^{***}	(0.0022)	$\Phi^{\mathbb{Q}}_{\sigma}$	1.0254^{***}	(0.007)
$\Phi_{y_1,\sigma}$	0	–	$\Phi^{\mathbb{Q}}_{y_1,\sigma}$	1.2678^{***}	(0.262)
$\Phi_{y_2,\sigma}$	0	–	$\Phi^{\mathbb{Q}}_{y_2,\sigma}$	-0.1903^{***}	(0.0434)
Φ_{π^*,y_1}	0	–	$\Phi^{\mathbb{Q}}_{\pi^*,y_1}$	0.0206^{***}	(0.0036)
Φ_{σ,y_1}	-0.0033^{***}	(0.001)	$\Phi^{\mathbb{Q}}_{\sigma,y_1}$	-0.0058^{***}	(0.0014)
Φ_{y_1}	0.9188^{***}	(0.0195)	$\Phi^{\mathbb{Q}}_{y_1}$	0.7499^{***}	(0.0105)
Φ_{y_2,y_1}	0	–	$\Phi^{\mathbb{Q}}_{y_2,y_1}$	0.0198^{***}	(0.0063)
Φ_{π^*,y_2}	0	–	$\Phi^{\mathbb{Q}}_{\pi^*,y_2}$	0.0148^{***}	(0.0035)
Φ_{σ,y_2}	-0.0034^{***}	$(9{\cdot}10^{-4})$	$\Phi^{\mathbb{Q}}_{\sigma,y_2}$	-0.0034^{***}	$(9{\cdot}10^{-4})$
Φ_{y_1,y_2}	-0.0852^{***}	(0.0194)	$\Phi^{\mathbb{Q}}_{y_1,y_2}$	-0.1634^{***}	(0.0208)
Φ_{y_2}	0.9791^{***}	(0.0033)	$\Phi^{\mathbb{Q}}_{y_2}$	0.9791^{***}	(0.0033)
Σ_{π^*}	0.1892^{***}	(0.02)	$\Sigma^{\mathbb{Q}}_{\pi^*}$	0.1892^{***}	(0.02)
Σ_{σ,π^*}	0	–	$\Sigma^{\mathbb{Q}}_{\sigma,\pi^*}$	0	–
Σ_{y_1,π^*}	0	–	$\Sigma^{\mathbb{Q}}_{y_1,\pi^*}$	0	–
Σ_{y_2,π^*}	0	–	$\Sigma^{\mathbb{Q}}_{y_2,\pi^*}$	0	–
Σ_{σ}	0.0083^{***}	(0.0021)	$\Sigma^{\mathbb{Q}}_{\sigma}$	0.0083^{***}	(0.0021)
$\Sigma_{y_1,\sigma}$	0	–	$\Sigma^{\mathbb{Q}}_{y_1,\sigma}$	0	–
$\Sigma_{y_2,\sigma}$	0	–	$\Sigma^{\mathbb{Q}}_{y_2,\sigma}$	0	–
Σ_{y_1}	1	–	$\Sigma^{\mathbb{Q}}_{y_1}$	1	–
Σ_{y_2,y_1}	0	–	$\Sigma^{\mathbb{Q}}_{y_2,y_1}$	0	–
Σ_{y_2}	1	–	$\Sigma^{\mathbb{Q}}_{y_2}$	1	–

Notes: Standard deviations are in parentheses and are calculated using the outer-product Hessian approximation. The '–' sign indicates that the parameter has been calibrated hence does not possess any standard deviation. Significance level: * <0.1, ** <0.05, *** <0.01.

Table 8: Parameter estimates: short-rate and the prices of risk: Standard QTSM

	r_t dynamics				
	estimates	std.		estimates	std.
α	0.0087***	(0.0013)	κ	0.1867***	(0.0134)
β_{π^*}	0	–	β_{y_1}	-0.0025***	$(1{\cdot}10^{-4})$
β_{σ}	-0.0122***	(0.0026)	β_{y_2}	0.001***	$(2{\cdot}10^{-4})$
$\underline{r} \cdot 1200$	0.0431***	(0.0103)	$\bar{\pi} \cdot 100$	3.1771***	(0.0946)
	Prices of risk and measurement errors standard deviations				
	estimates	std.		estimates	std.
λ_{0,π^*}	0	–	λ_{0,y_1}	0	–
$\lambda_{0,\sigma}$	0	–	λ_{0,y_2}	0	–
λ_{1,π^*} 0.5001***	(0.1027)		λ_{1,π^*,y_1} 0.109***	(0.0232)	
λ_{1,σ,π^*}	0	–	λ_{1,σ,y_1}	-0.3055***	(0.1203)
λ_{1,y_1,π^*}	0	–	λ_{1,y_1}	-0.169***	(0.0211)
λ_{1,y_2,π^*}	-0.0378***	(0.0027)	λ_{1,y_2,y_1}	0.0198***	(0.0063)
$\lambda_{1,\pi^*,\sigma}$	0	–	λ_{1,π^*,y_2}	0.0781***	(0.0213)
$\lambda_{1,\sigma}$	4.024***	(1.5329)	λ_{1,σ,y_2}	0	–
$\lambda_{1,y_1,\sigma}$	1.2678***	(0.262)	λ_{1,y_1,y_2}	-0.0782***	(0.0213)
$\lambda_{1,y_2,\sigma}$	-0.1903***	(0.0434)	λ_{1,y_2}	0	–
λ_r	0	–			
σ_R	0.0617***	$(9{\cdot}10^{-4})$	σ_R^*	0.1262***	(0.0034)
$\sigma_{\pi}^{(12)}$	0.509	–	$\sigma_{\pi}^{(120)}$	0.389	–
$\sigma_{S_R}^{(3)}$	0.231	–	$\sigma_{S_R}^{(12)}$	0.422	–
σ_{ZLB}	0.0522	–			

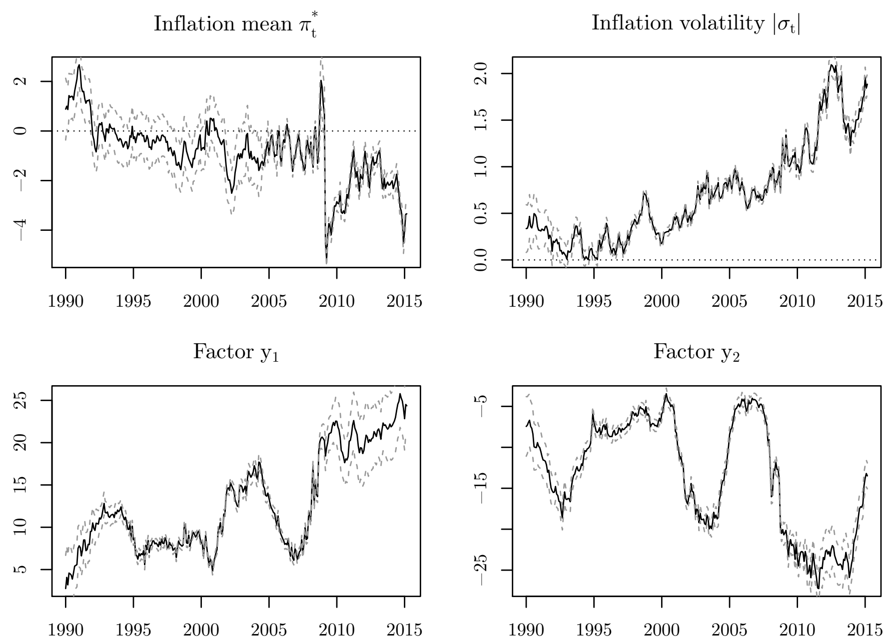
Notes: Standard deviations are in parentheses and are calculated using the outer-product Hessian approximation. The '–' sign indicates that the parameter has been calibrated hence does not possess any standard deviation. Significance level: * <0.1, ** <0.05, *** <0.01.

Table 9: Model fit and characteristics: standard QTSM

Maturities (months)	1	12	24	36	60	84	120
Nominal rates RMSE (bps)	8.09	8.28	4.93	6.25	5.74	3.83	6.08
Real rates RMSE (bps)	-	17.87	7.76	7.94	11.66	10.67	9.42
Probabilities (in %)	$\mathbb{P}(r_t < 0.25\%) = 9.45$						

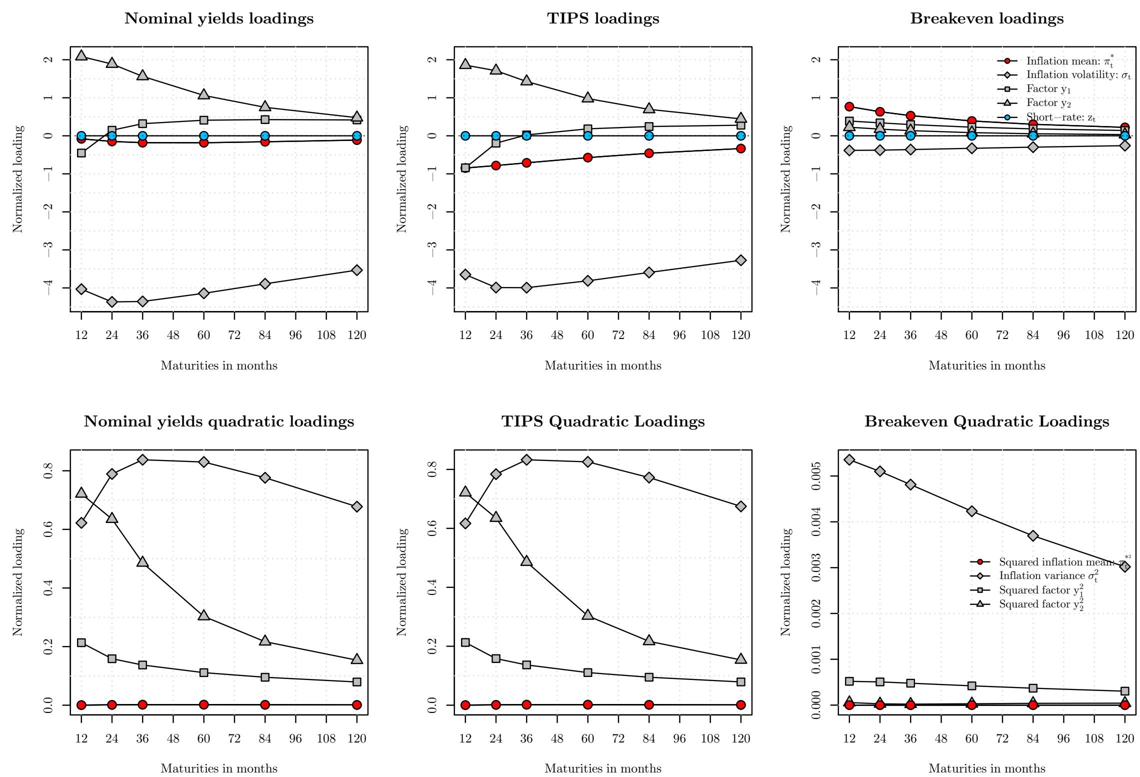
Note: Probabilities are calculated with simulated paths of length 100,000.

Figure 14: Filtered factors: standard QTSM



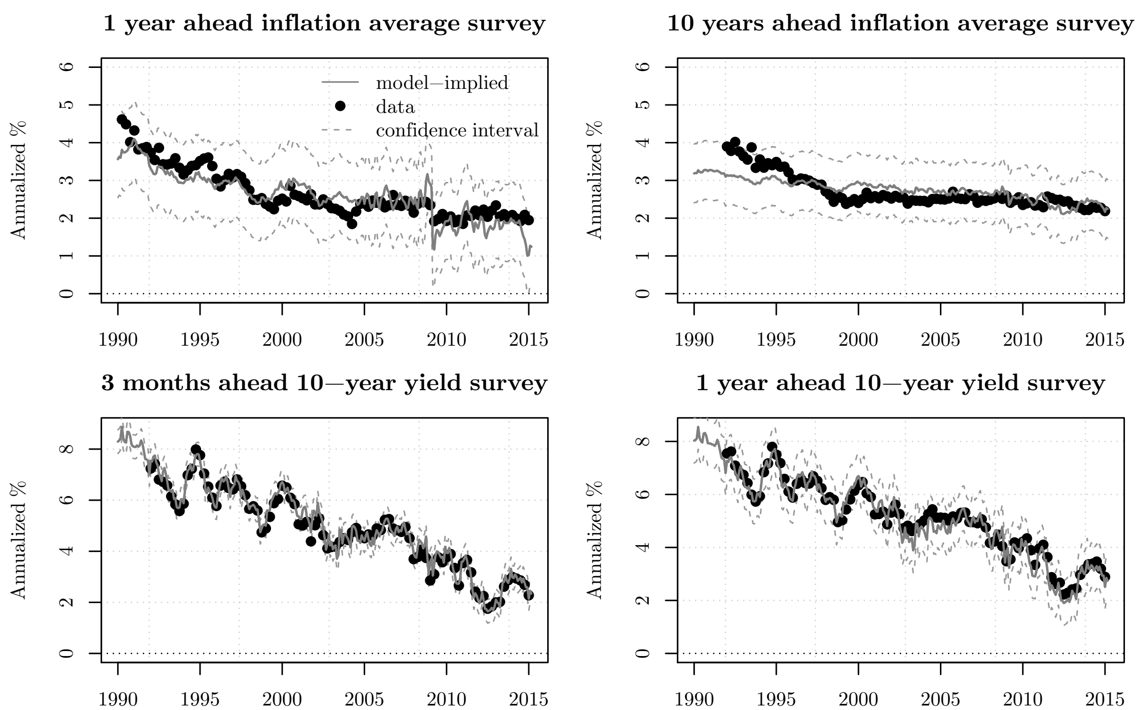
Notes: The unit for inflation central tendency π_t^* and inflation volatility σ_t is in percentage points. Grey dashed lines are 95% confidence bands. The red vertical line delimits the beginning of the zero lower bound period.

Figure 15: Factors loadings of yields: standard QTSM



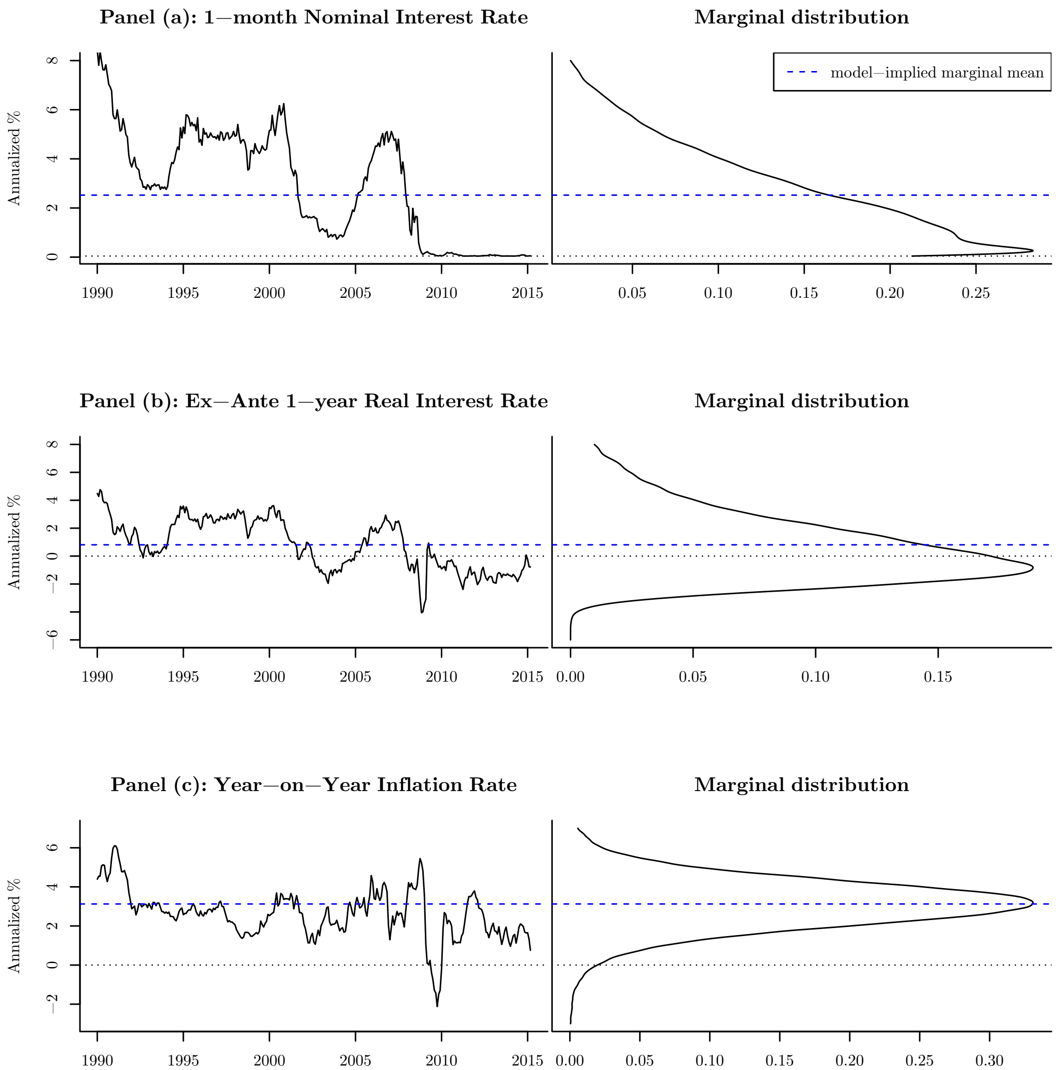
Notes: This plot gathers the linear loadings of the nominal interest rates, of the real rates, and the quadratic loadings with respect to maturity. These loadings are normalized by the in-sample standard deviation of the corresponding filtered factor to be comparable with each other.

Figure 16: Fitted series of survey data: standard QTSM



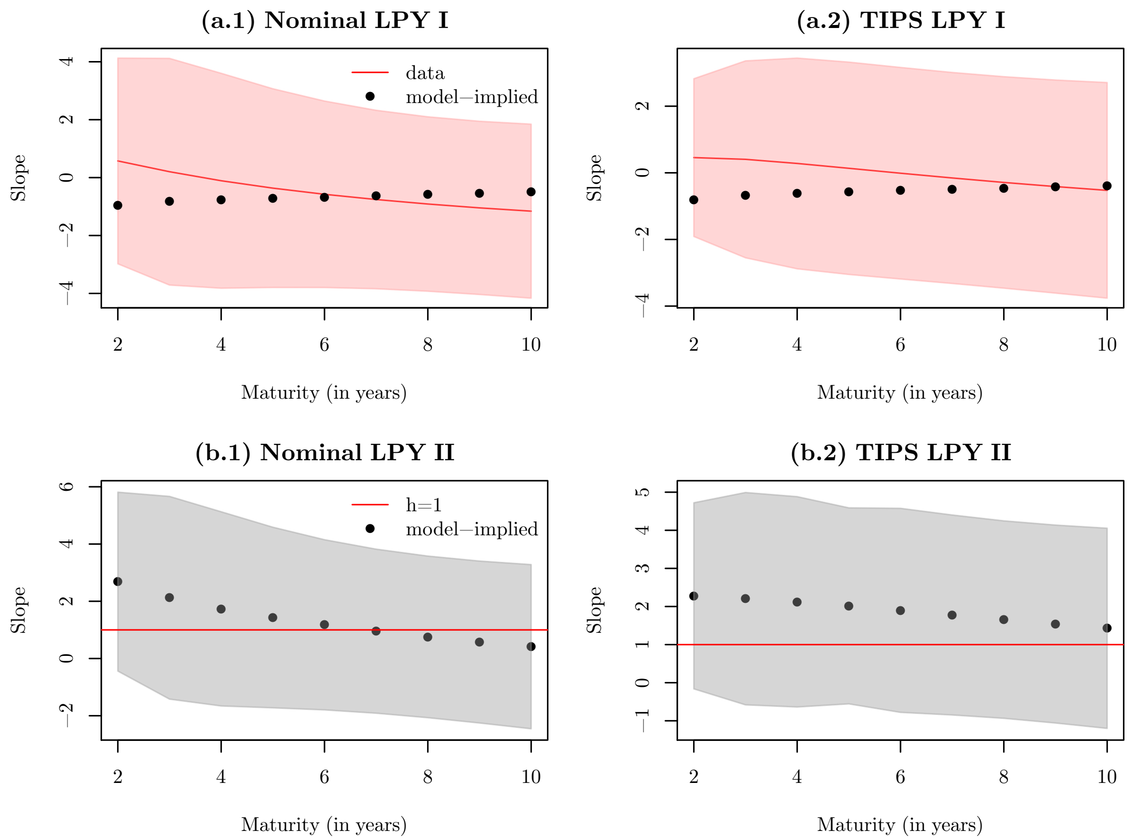
Notes: The black dots correspond to observed forecast data. The grey solid lines correspond to the model-implied forecasted values. Top graphs correspond respectively to the one-year ahead and 10-year ahead inflation average surveys. Medium graphs correspond respectively to the three-months ahead and one-year ahead 10-year yield survey. Units are in annualized percentage points. Bottom graphs correspond respectively to the fitted natural logarithm of ELB probabilities, and of the exponential of the latter. Confidence intervals computed using the measurement errors standard deviations are plotted in grey dashed lines. The red vertical line delimits the beginning of the zero lower bound period.

Figure 17: Distribution of short-term interest rates and inflation: standard QTSM



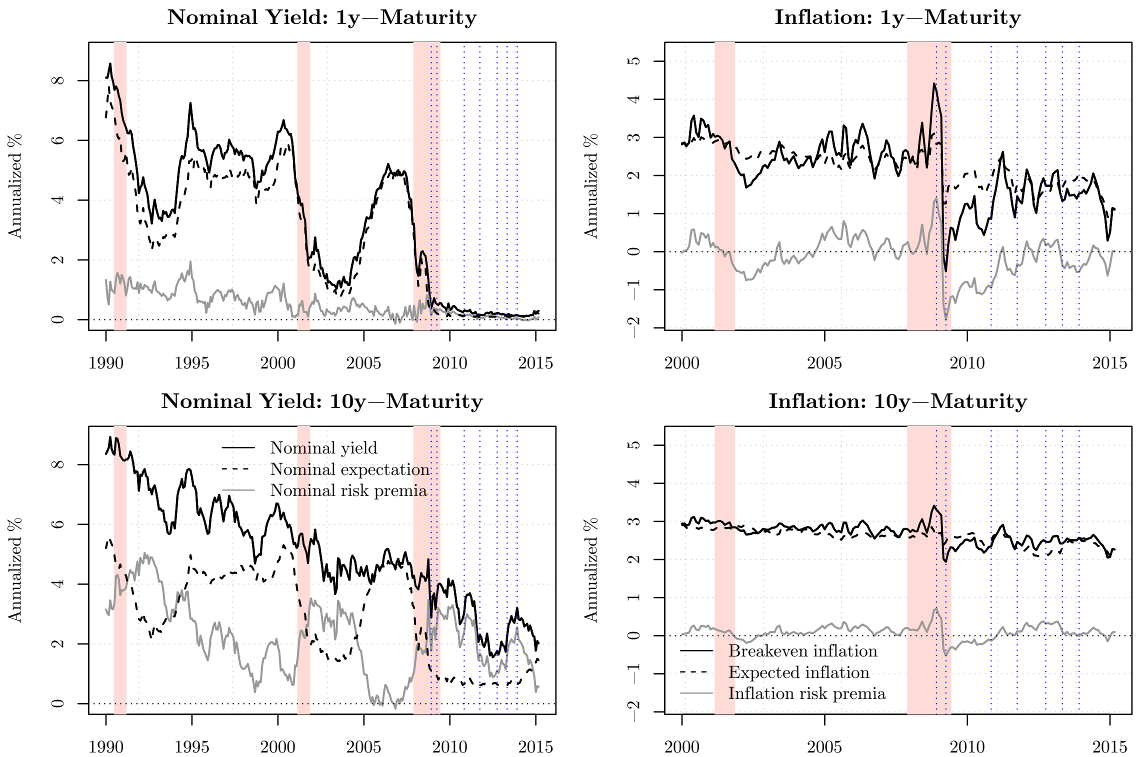
Notes: Marginal distributions on the right-hand side are compute with kernel density estimates on 100,000 simulations.

Figure 18: Campbell-Shiller regression slopes: standard QTSM



Notes: These graphs present the slopes of Campbell and Shiller regressions with a 12-months holding period. On panel (a), the red solid line gathers the slope estimates obtained with yields data. 95% Confidence intervals are computed using Newey-West robust estimators with automatically selected lag and are indicated with the red-shaded areas. Model-implied estimates are indicated with the black dots and computed with the yields and inflation expectation and variance formulas. On panel (b), the red solid line represents the theoretical values of the regression, namely one for all maturities. Model-implied estimates are indicated with the black dots and computed performing the Campbell and Shiller regressions where the dependent variable is adjusted by the model-implied expected excess returns. 95% Confidence intervals are computed using Newey-West robust estimators with automatically selected lag and are indicated with the grey-shaded areas.

Figure 19: Decomposition of interest rates: standard QTSM



Notes: The first column presents results for the nominal yields components, whereas the second column presents results for the inflation components. The first row presents to the observed data (black solid line), the risk premia (grey solid line), and the expected component (black dashed line) at the one year maturity. The second row presents the same components at the 10 year maturity. Units are in annualized percentage points. The red vertical line delimits the beginning of the zero lower bound period. Pink shaded areas are NBER recession periods. The blue dashed lines are the different unconventional monetary policy episodes, namely: QE1, QE1-extension, QE2, Operation Twist, QE3, Taper tantrum, and the Tapering.